Exchange economy and multiple equilibriums

Consider an exchange economy with two agents whose preferences are represented by the utility functions

$$u_1(x_1^1, x_1^2) = x_1^1 - \frac{1}{8}(x_1^2)^{-8}$$

$$u_2(x_2^1, x_2^2) = -\frac{1}{8}(x_2^1)^{-8} + x_2^2$$

Their initial endowments are $\omega^1 = (2, r)$, $\omega^2 = (r, 2)$, where r is momentarily a parameter. Find the demand functions. Where r is chosen so that the demand for each good is positive. Prove that the equilibrium prices are determined by the equation

$$\left(\frac{p_2}{p_1}\right)^{-1/9} + r\left(\frac{p_1}{p_2}\right)^{8/9} - \left(\frac{p_1}{p_2}\right) = r$$

show that for the value $r = 2^{8/9} - 2^{1/9}$ there are three solutions

$$\frac{p_1}{p_2} = 1, 2, \frac{1}{2}$$

and therefore there are three competitive equilibria.

Solution

The demand function of agent 1 is the solution to the maximization problem

$$\max x - \frac{1}{8}y^{-8}$$
$$\text{s.t. } p_1 x + p_2 y = r^1$$

where $r^1 = 2p_1 + p_2 r$. The associated Lagrangian is

$$L = x - \frac{1}{8}y^{-8} + \lambda(r^1 - p_1 x - p_2 y)$$

The first-order conditions are

$$\frac{\partial L}{\partial x} = 1 - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial y} = y^{-9} - \lambda p_2 = 0$$

Therefore

$$\lambda = \frac{1}{p_1}$$
$$\lambda = \frac{y^{-9}}{p_2}$$

Combining λ

$$\frac{1}{p_1} = \frac{y^{-9}}{p_2}$$
$$y = \left(\frac{p_2}{p_1}\right)^{-1/9}$$

and using the budget constraint we see that

$$x = \frac{r^1}{p_1} - \frac{p_2 y}{p_1}$$
$$x = 2 + r\frac{p_2}{p_1} - \left(\frac{p_2}{p_1}\right)^{8/9}$$

This means:

$$x_1^1 = 2 + r\frac{p_2}{p_1} - \left(\frac{p_2}{p_1}\right)^{8/9}$$
$$x_2^1 = \left(\frac{p_2}{p_1}\right)^{-1/9}$$

Similarly, we obtain the demand of agent 2 the lagrangian is

$$L = y - \frac{1}{8}x^{-8} + \lambda(r^2 - p_1 x - p_2 y)$$

The first-order conditions are

$$\frac{\partial L}{\partial y} = 1 - \lambda p_2 = 0$$

$$\frac{\partial L}{\partial x} = x^{-9} - \lambda p_1 = 0$$

Therefore

$$\lambda = \frac{1}{p_2}$$

$$\lambda = \frac{x^{-9}}{p_1}$$

Combining λ

$$\frac{1}{p_2} = \frac{x^{-9}}{p_1}$$

$$x = \left(\frac{p_1}{p_2}\right)^{-1/9}$$

and using the budget constraint we see that

$$y = \frac{r^2}{p_2} - \frac{p_1 x}{p_2}$$

$$y = 2 + r\frac{p_1}{p_2} - \left(\frac{p_1}{p_2}\right)^{8/9}$$

This means:

$$x_2^2 = r + 2\frac{p_1}{p_2} - \left(\frac{p_2}{p_1}\right)^{8/9}$$

$$x_1^2 = \left(\frac{p_1}{p_2}\right)^{-1/9}$$

To calculate the equilibrium prices, we use the market clearing condition for good 2:

$$2 + r = x_2^1 + x_1^1$$

$$2 + r = 2 + r\frac{p_1}{p_2} - \frac{p_1}{p_2}^{8/9} + \left(\frac{p_2}{p_1}\right)^{-1/9}$$

$$r = r\frac{p_1}{p_2} - \frac{p_1}{p_2}^{8/9} + \left(\frac{p_2}{p_1}\right)^{-1/9}$$

If $r = 2^{8/9} - 2^{1/9}$ we obtain

$$2^{8/9} - 2^{1/9} = (2^{8/9} - 2^{1/9})\frac{p_1}{p_2} - \frac{p_1}{p_2}^{8/9} + \left(\frac{p_2}{p_1}\right)^{-1/9}$$

Clearly, $p_1/p_2 = 1$ is a solution:

$$2^{8/9} - 2^{1/9} = (2^{8/9} - 2^{1/9}) - 1 + 1$$

Now suppose $p_1/p_2 = 2$. In that case,

$$2^{8/9} - 2^{1/9} = 2(2^{8/9} - 2^{1/9}) - 2^{8/9} + 2^{1/9}$$

$$2^{8/9} - 2^{1/9} = 2^{8/9} - 2^{1/9}$$

and substituting for $p_1/p_2 = 1/2$:

$$2^{8/9} - 2^{1/9} = (1/2)(2^{8/9} - 2^{1/9}) - (1/2)^{8/9} + (2)^{-1/9}$$

$$0.772 = 0.772$$